International Journal of Theoretical Physics, Vol. 44, No. 9, September 2005 (© 2005)
DOI: 10.1007/s10773-005-4834-3

Quantum States of a Trapped Dirac Particle in a Pseudoscalar Potential

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Received March 7, 2005; accepted March 31, 2005

We obtain the quantum states of a trapped Dirac particle in the presence of a pseudoscalar potential. A change in the geometrical boundary condition can cause an effective electromagnetic field which can act on the trapped object. The nonrelativistic limit is discussed.

KEY WORDS: pseudoscalar potential; Dirac equation; boundary effect.

1. INTRODUCTION

In quantum physics, many works (Meekhof *et al.*, 1996; Monroe *et al.*, 1996; Shore and Knight, 1993) have discussed the properties of particles in the presence of interacting potentials with different internal and external structures. In 1992, Shishkin and Villalba (Shishkin and Villalba, 1992) studied the quantum states of Dirac particles by applying the algebraic method of variable separation of the Dirac equation in the presence of pseudoscalar potential, of which the solutions were obtained by Villalba lately (Villalba, 1997; Villalba and Ramiro, 2003). They emphasized the pseudoscalar potential with a dependence inversely proportional to the radial variable does not bind particle. However, things are different when Dirac particles are confined by a macroscopic or harmonic trap, since their bound states are then deeply connected with the geometrical boundary effect (Heinzen and Wineland, 1990; Orzel *et al.*, 2001; Wineland *et al.*, 1994). Therefore, people believe that a valid analysis of these problems would be useful for understanding the dynamical evolution of the quantum system, for example, the boundary effect and decoherence (Pereshogin and Pronin, 1991; Zurek, 1991).

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0020-7748/05/0900-1621/0

The purpose of the present paper is to study the quantum state of a Dirac particle bounded by a time-dependent infinite spherical well in pseudoscalar potential. In Section 2 we solve the Dirac equation of the trapped system. In Section 3 the effective Hamiltonian of the trapped system with a variable boundary is constructed. In Section 4 we compute the system energy spectrum by solving the nonrelativistic form of the Dirac equation. Some concluding remarks are presented in Section 5.

2. DIRAC EQUATION FOR TRAPPED PARTICLE

To study the quantum states of the trapped system, we here consider a very simple situation in that a Dirac particle in pseudoscalar potential is trapped by a spherical well of range $a_\circ$, like a hydrogen atom bounded. The pseudoscalar potential function reads

$$V(r; a_\circ) = \begin{cases} \gamma^5 \frac{\alpha}{r}, \ r \leq a_\circ, \ \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 \\ \infty, \quad r > a_\circ \end{cases} \tag{1}$$

where α is a real constant and the standard Dirac matrices are given in terms of Pauli matrices as follows

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \tag{2}$$

As $\alpha = 0$, $V(r; a_\circ)$ becomes the usual infinite spherically symmetric well, for which $V(r) = 0$ as $r \leq a_\circ$ and ∞ otherwise. So that, the particle can be considered to be confined to the inner part of the well, far away from the wall.

Now, we use the spherical polar coordinates (r, θ, ϕ) in natural units ($\hbar = c = 1$), and express Dirac equation of the trapped particle in the rotating diagonal tetrad

$$\gamma^0 \partial_t \Psi = \hat{H}_\circ \Psi \tag{3}$$

with the Hamiltonian

$$\hat{H}_\circ = -\gamma^1 \partial_r - \frac{\gamma^2}{r} \partial_\theta - \frac{\gamma^3}{r \sin\theta} \partial_\varphi - i V(r) - m \tag{4}$$

endowed with boundary condition $\Psi(r = a_\circ, t) = 0$. Here, the insertion of the imaginary unit i before $V(r)$ is to make the Dirac Hamiltonian Hermitian. Consider the potential $V(r)$ does not depend on time, nor on the angles θ and φ, we write Ψ in the following form (Villalba, 1997)

$$\Psi(r, \theta, \varphi, t) = \mathbf{S}^{-1} \gamma^0 \gamma^1 \Phi(r, t) \Theta(\theta, \varphi) \tag{5}$$

S represents the following similarity transformation

$$\mathbf{S} = \frac{\gamma^1\gamma^2 + \gamma^2\gamma^3 + \gamma^3\gamma^1 + 1}{2r\sqrt{\sin\theta}} \exp\left[-\frac{\varphi}{2}\gamma^1\gamma^2 - \frac{\theta}{2}\gamma^3\gamma^1\right] \tag{6}$$

This transformation can help us to eliminate the contribution of the spinor connection and the coordinate dependence of γ matrices. Then, by applying the algebraic method to separate the variables in Eq. (3), we arrive at

$$\left[(\gamma^0\partial_t + \gamma^1\partial_r + m)\gamma^0\gamma^1 + \frac{il}{r}\right]\Phi(r,t) = 0 \tag{7}$$

$$(\hat{L}_{\mathrm{B}} + \alpha\gamma^2\gamma^3)\Theta(\theta,\varphi) = l\Theta(\theta,\varphi) \tag{8}$$

where the separating constant l may be conventionally treated as the quantum number of the orbital angular momentum, and the Brill angular-momentum operator $\hat{L}_{\mathrm{B}}$ takes the form of (Brill and Wheeler, 1957)

$$\hat{L}_{\mathrm{B}} = -i\left(\gamma^2\partial_\theta + \frac{\gamma^3}{\sin\theta}\partial_\varphi\right)\gamma^0\gamma^1 \tag{9}$$

the operator having the eigenvalue of $L_{\mathrm{B}}^2 = l^2 - \alpha^2$, anticommutes with the matrix $\gamma^2\gamma^3$, i.e. $\{\hat{L}_{\mathrm{B}}, \gamma^2\gamma^3\}_+$. Right now, it should be noticed that, the contribution of the pseudoscalar potential has gone into Eq. (8) due to its particular dependence on radial variable.

Further, Eq. (7) can be written in the bispinor form

$$\left[\frac{d^2}{dr^2} - \frac{l(l\pm 1)}{r^2}) + (E^2(a_\circ) - m^2)\right]\begin{pmatrix}\Phi_+\\ \Phi_-\end{pmatrix} \tag{10}$$

where Φ_+ and Φ_- represent the upper and lower components of the bispinor, respectively, and $E(a_\circ)$ is the eigenvalue for $\hat{H}_\circ$. It is clear that, the last equation describes nothing but the "free" states for the relativistic Dirac electron in pseudoscalar field (1), whose solutions can be expressed in terms of Bessel functions as (Bransden and Joachain, 1989; Villalba, 1997)

$$\Phi_\pm = C_{l\pm\frac{1}{2}}(a_\circ)\sqrt{r}J_{l\pm\frac{1}{2}}(k(a_\circ)r),\, k(a_\circ) = \sqrt{E^2(a_\circ) - m^2} \tag{11}$$

$C_{l\pm\frac{1}{2}}(a_\circ)$ is the normalization constant, $k(a_\circ)$ the wave number dependent of the potential well scale $a_\circ$, and $J_{l\pm\frac{1}{2}}(kr)$ the spherical Bessel functions.

In the case that the potential range is variable, namely $a_\circ \to a = a(t)$, we must solve Eq. (3) subject to the time-dependent boundary condition, $\Phi(r = a(t), t) = 0$. Then, the normalized solutions take the time-dependent form as

$$\Psi = \mathbf{S}^{-1}\gamma^0\gamma^1\Phi(r)\Theta(\theta,\varphi) \times \exp\left[-i\int_0^t E(a(t'))\,dt'\right] \tag{12}$$

with

$$\Phi(r) = \begin{pmatrix} C_{l+1/2}(a)\sqrt{r}J_{l+1/2}(kr) \\ C_{l-1/2}(a)\sqrt{r}J_{l-1/2}(kr) \end{pmatrix} \tag{13}$$

the wave number $k(a)$ is defined by the boundary condition $J_{l\pm\frac{1}{2}}(ka) = 0$, namely

$$k(a) = \sqrt{E^2(a) - m^2} = \frac{\chi_{n_r l}}{a} \tag{14}$$

$\chi_{n_r l}$ denotes the n_rth solution of the spherical Bessel function of order l, n_r the radial quantum number. Correspondingly, the normalization constants $C_{l\pm 1/2}(a)$ are given by

$$C_{l\pm 1/2} = \left[\frac{-2}{a^3 J_{l\pm 1/2-3/2}(\chi_{n_r l}) J_{l\pm 1/2+1/2}(\chi_{n_r l})} \right]^{1/2} \tag{15}$$

In the following section, we will use these relations to construct the effective Hamiltonian of the system as the variable boundary condition is considered.

3. EFFECTIVE HAMILTONIAN

To obtain the effective system Hamiltonian, we need to analyze the action of the operator $\partial/\partial t$ on the wave functions $\Psi(t; a)$. The naive understanding of the partial derivative is that (Pereshogin and Pronin, 1991)

$$\gamma^0 \partial_t \Psi(t; a) = \{E(a) + \gamma^0 \dot{a} \partial_a\} \Psi(t; a), \qquad \partial_a = \begin{pmatrix} \partial_{a+} & 0 \\ 0 & \partial_{a-} \end{pmatrix} \tag{16}$$

Note the fact that $C_{l\pm\frac{1}{2}}(a)$ is a function of a and $J_{l\pm\frac{1}{2}}(kr)$ depend on it through the wave number $k(a)$, we can clarify the a derivative operator ∂_a by the following

$$\partial_{a\pm} = \frac{\partial ln C_{l\pm\frac{1}{2}}}{\partial a} + \frac{\partial k}{\partial a} \frac{\partial ln J_{l\pm\frac{1}{2}}}{\partial k} \tag{17}$$

In the spherical polar coordinate, we have the operator

$$\hat{r} \to i \frac{1}{k} \frac{\partial}{\partial k} k \tag{18}$$

of which, the result acting on the Bessel functions, $\hat{r} J_{l\pm\frac{1}{2}}(kr) = r J_{l\pm\frac{1}{2}}(kr)$, suggests

$$\partial_{a\pm} = \frac{\partial ln(C_{l\pm\frac{1}{2}}/k)}{\partial a} - ir \frac{\partial k}{\partial a} = \frac{1}{2a} + ir \frac{\chi_{n_r l}}{a^2} \tag{19}$$

here Eqs. (14) and (15) and the relation $\frac{\partial}{\partial k} = \frac{1}{k}(\frac{\partial}{\partial k}k - 1)$ are used. In practice, the last term in Eq. (19) can be expressed as

$$ir\frac{\chi_{n_r l}}{a^2} = i\frac{e}{\dot{a}}\mathbf{e}_{\theta,\varphi} \cdot \mathbf{A}, \mathbf{A} = \frac{\chi_{n_r l}\dot{a}}{ea^2} r\mathbf{e}_{\theta,\varphi} \tag{20}$$

$\mathbf{e}_{\theta,\varphi}$ represents the unit vector along the spherical direction, meaning the defined vector $\mathbf{A}$ only has the spherical component(i.e., $A_r = 0$) (Pereshogin and Pronin, 1991; Qian-Kai, 1999). So we arrive at

$$\partial_{a\pm} = \frac{1}{2a} + i\frac{e}{\dot{a}}\mathbf{e}_{\theta,\varphi} \cdot \mathbf{A} \tag{21}$$

In the Dirac matrices' representation, we are allowed to decompose $\gamma^0\partial_a$ through a connection form in the infinitesimal one

$$\gamma^0\partial_a = -\gamma^0\frac{1}{2a} + i\frac{e}{\dot{a}}\overrightarrow{\gamma} \cdot \mathbf{A}, \overrightarrow{\gamma} = (\gamma^1, \gamma^2, \gamma^3) \tag{22}$$

by which, the effective Hamiltonian can be constructed as

$$\hat{H}_{\text{eff}} = \hat{H}_0 + i\gamma^\mu e A_\mu \tag{23}$$

with the four-potential A_μ

$$A_\mu = (\phi, \mathbf{A}), \phi = -i\frac{\dot{a}}{2ae} \tag{24}$$

it reflects the interaction between the outer electron and electromagnetic field caused by the boundary motion. When the stationary condition, $\dot{a} = 0$ is met, the effective Hamiltonian will come back to the usual one, i.e., $\hat{H}_{\text{eff}}(\dot{a} \to 0) \to \hat{H}_\circ$. So we emphasize that the trapped Dirac particle with a variable boundary is equivalent to the system interacting with the induced electromagnetic field.

4. NONRELATIVISTIC LIMIT

Right now, the ordinary time derivative in Dirac equation reads

$$\gamma^0\partial_t\Psi = \hat{H}_{\text{eff}}\Psi \tag{25}$$

We solve this equation by clarifying the determination of the spherical vector potential $\mathbf{A}$ as

$$\mathbf{A} = \frac{\chi_{n_r l}\dot{a}}{ea^2} r\mathbf{e}_{\theta,\phi} = \frac{\chi_{n_r l}\dot{a}}{ea^2}(\mathbf{e}_l \times \mathbf{r}) \tag{26}$$

$\mathbf{e}_l$ denotes the unit vector along the direction normal to $\mathbf{r}$ and $\mathbf{e}_{\theta,\varphi}$, which is parallel to the angular momentum vector $\mathbf{L}$(i.e., $\mathbf{e}_l = \mathbf{L}/|L|$) for $l \neq 0$. Consider that $\mathbf{A}$

can be expressed in terms of the magnetic field $\mathbf{B}$ as $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$, then we have

$$\mathbf{B} = \frac{2\chi_{n_r l}\dot{a}}{ea^2}\mathbf{e}_l \tag{27}$$

It clearly suggests that, as $\dot{a} > 0$ or $\dot{a} < 0$, the induced magnetic field $\mathbf{B}$ of length $2\dot{a}\chi_{n_r l}/(ea^2)$ should be parallel or antiparallel to $\mathbf{L}$, and therefore appears a similar feature to it.

Substituting the effective Hamiltonian (23) followed by Eq. (24) into Eq. (25), we readily obtain the following form

$$\left[E + \gamma^1\partial_r + \frac{\gamma^2}{r}(\partial_\theta + eA_\theta) + \frac{\gamma^3}{r\sin\theta}(\partial_\varphi + eA_\varphi)\right]\Psi = 0 \tag{28}$$

In the nonrelativistic limit, the equation comparing with Eq. (7), takes the Pauli's form (Villalba and Ramiro, 2003)

$$\left[\frac{1}{2m}\left(\frac{d^2}{dr^2} - \frac{l(l\pm 1)}{r^2}\right) + \left(E' - \frac{e\mathbf{L}\cdot\mathbf{B}}{2m} - \frac{e\mathbf{s}\cdot\mathbf{B}}{m} - \frac{e^2B^2r^2}{8m}\right)\right]\begin{pmatrix}\Phi_+\\ \Phi_-\end{pmatrix} = 0 \tag{29}$$

where $E'(a)(= E(a) - m)$ is the nonrelativistic energy of the trapped particle and $\mathbf{s}$ its spin. In the case of the slow wall motion, the term proportional to $B^2(\propto \dot{a}^2 \sim 0)$ may be omitted, and then Eq. (29) becomes

$$\left[\frac{1}{2m}\left(\frac{d^2}{dr^2} - \frac{l(l\pm 1)}{r^2}\right) + (E' + W_l + W_s)\right]\begin{pmatrix}\Phi_+\\ \Phi_-\end{pmatrix} = 0 \tag{30}$$

followed by

$$\begin{cases} W_l = -\vec{\mu}_l\cdot\mathbf{B}, & \vec{\mu}_l = \frac{e}{2m}\mathbf{L}\\ W_s = -\vec{\mu}_s\cdot\mathbf{B}, & \vec{\mu}_s = \frac{e}{m}\mathbf{s}\end{cases} \tag{31}$$

the two, respectively, reflect the energies of the orbital and spin angular magnetic moments($\vec{\mu}_l$, $\vec{\mu}_s$) coupling with $\mathbf{B}$. So that, the total energy of the trapped object is

$$E'_{\text{tot}} = E' + \Delta E_{l+s},\ \Delta E_{l+s} = W_l + W_s \tag{32}$$

This means the variable boundary condition can cause the energy shift ΔE that is dependent on $\dot{a}$. In the stationary case, i.e., $\dot{a} \to 0$, W_l and W_s both tend to zero.

To take into account how the orbital and spin magnetic moments couple to the induced magnetic field in the Pauli equation, we compute the values of W_l and W_s according to the model of the angular momentum, those are

$$\begin{cases} W_l = -\frac{\chi_{n_r l}\dot{a}}{ma^2}\sqrt{l(l+1)},\\ W_{\pm 1/2} = -\frac{\chi_{n_r l}\dot{a}}{ma^2}\cos\vartheta_{\pm 1/2},\end{cases} \quad \text{for } l \neq 0 \tag{33}$$

with

$$\cos\vartheta_{\pm 1/2} = \pm\left(\frac{l}{l+1}\right)^{\pm 1/2} \tag{34}$$

$\vartheta_{+1/2}$, $\vartheta_{-1/2}$ represent the angles between **B** and $\vec{\mu}_s$ for $s = \frac{1}{2}$ and $s = -\frac{1}{2}$ respectively. In the case of $l = 0$, the obtained results are

$$\begin{cases} W_l = 0, \\ W_{\pm 1/2} = \mp\frac{\chi_{n_r l}\dot{a}}{ma^2}, \end{cases} \quad \text{for } l = 0 \tag{35}$$

In this way, we present the energy shift of the trapped particle that is engendered by the variable boundary condition. As an example for the $2P_{3/2}$ level, it has the value of

$$\Delta E_{3/2} = W_1 + W_{+1/2} = -\frac{3\sqrt{2}\chi_{n_r l}\dot{a}}{2ma^2} \tag{36}$$

5. CONCLUSION

By Dirac equation, we investigate the quantum state of trapped system consisting of a Dirac particle in the presence of pseudoscalar potential. Under the nonstationary boundary condition, the effective system Hamiltonian is constructed, which represents the interaction between the trapped system and the induced electromagnetic field. After clarifying the determination of the electromagnetic field, we obtain the nonrelativistic limit of Dirac equation in the Pauli's form, and compute its energy spectrum. It shows that the variable trapped boundary can cause the energy level shift of the system.

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